Exercise 1: The strain energy density is given by,

$$W(\varepsilon_{ij},\sigma_{ij}) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \qquad \text{(a)}$$

Use the Hook's law to express the energy density

(1) in terms of stresses,

(2) in terms of strains,

(3) take the derivative of the energy with respect to strain, respectively, stress to obtain the stresses and strains.

Solution[1]

1. Hooks law: $\varepsilon_{ij} = \frac{1}{E}\left[(1+v)\sigma_{ij} - v\delta_{ij}\sigma_{nn}\right]$ (b)

Insert (b) in (a),

$$W^*(\sigma_{ij}) = \frac{1}{2}\frac{1}{E}\left[(1+v)\sigma_{ij} - v\delta_{ij}\sigma_{nn}\right]\sigma_{ij} = \frac{1}{2E}\left[(1+v)\sigma_{ij}\sigma_{ij} - v\delta_{ij}\sigma_{ij}\sigma_{nn}\right]$$
$$= \frac{1+v}{2E}\sigma_{ij}\sigma_{ij} - \frac{v}{2E}\sigma_{ii}\sigma_{nn}$$

2. Hook's law: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ (c)

Insert (c) in (a)

$$W(\varepsilon_{ij}) = \frac{1}{2}\left(\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\right)\varepsilon_{ij} = \frac{1}{2}\lambda\varepsilon_{kk}\delta_{ij}\varepsilon_{ij} + \mu\varepsilon_{ij}\varepsilon_{ij} = \frac{1}{2}\lambda\varepsilon_{kk}\varepsilon_{ii} + \mu\varepsilon_{ij}\varepsilon_{ij}.$$

3. $$\sigma_{mn} = \frac{\partial W(\varepsilon_{ij})}{\partial\varepsilon_{mn}} = \frac{1}{2}\lambda\left(\frac{\partial\varepsilon_{ii}}{\partial\varepsilon_{mn}}\varepsilon_{jj} + \varepsilon_{ii}\frac{\partial\varepsilon_{jj}}{\partial\varepsilon_{mn}}\right) + \mu\left(\frac{\partial\varepsilon_{ij}}{\partial\varepsilon_{mn}}\varepsilon_{ij} + \varepsilon_{ij}\frac{\partial\varepsilon_{ij}}{\partial\varepsilon_{mn}}\right)$$
$$= \frac{1}{2}\lambda\left(\delta_{im}\delta_{in}\varepsilon_{jj} + \varepsilon_{ii}\delta_{jm}\delta_{jn}\right) + 2\mu\left(\delta_{im}\delta_{jn}\varepsilon_{ij}\right)$$
$$= \frac{1}{2}\lambda\left(\delta_{mn}\varepsilon_{jj} + \varepsilon_{ii}\delta_{mn}\right) + 2\mu\varepsilon_{mn}$$

$\Rightarrow \lambda\delta_{mn}\varepsilon_{jj} + 2\mu\varepsilon_{mn} = \sigma_{mn}$ which is relation (c).

Similarly,

[1] We indicate the strain energy in terms of stresses as $W^*(\boldsymbol{\sigma})$ to distinguish it from the strain energy $W(\boldsymbol{\varepsilon})$ expressed in terms of strains.

$$\varepsilon_{kl} = \frac{\partial W^*(\sigma_{ij})}{\partial \sigma_{kl}} = \frac{1+v}{2E}\left(2\sigma_{ij}\frac{\partial \sigma_{ij}}{\partial \sigma_{kl}}\right) - \frac{v}{2E}\left(\frac{\partial \sigma_{ii}}{\partial \sigma_{kl}}\sigma_{nn} + \sigma_{ii}\frac{\partial \sigma_{nn}}{\partial \sigma_{kl}}\right)$$

$$= \frac{1+v}{E}\sigma_{ij}\delta_{ik}\delta_{jl} - \frac{v}{2E}\left(\sigma_{nn}\delta_{ik}\delta_{il} + \sigma_{ii}\delta_{nk}\delta_{nl}\right)$$

$$= \frac{1+v}{E}\sigma_{kl} - \frac{v}{2E}\left(\sigma_{nn}\delta_{kl} + \sigma_{ii}\delta_{kl}\right) = \frac{1+v}{E}\sigma_{kl} - \frac{v}{E}\sigma_{nn}\delta_{kl}$$

which is relation (b).

Exercise 2: Consider the Hook's law in index form (numbers refer to the book by Botsis & Deville and are used here for convenience),

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \qquad (6.174)$$

and the following definitions for the deviatoric stresses and strains:

$$\sigma_{ij} = \sigma_{ij}^{d} + \sigma_0 \delta_{ij}, \quad \sigma_0 = \frac{1}{3}\sigma_{kk}, \quad \sigma_{ij}^{d} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad (B6.175)$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{d} + \varepsilon_0 \delta_{ij}, \quad \varepsilon_0 = \frac{1}{3}\varepsilon_{kk}, \quad \varepsilon_{ij}^{d} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \qquad (B6.176)$$

1. Show that (a) is equivalent to

    a. $\sigma_{ij}^{d} = 2\mu \varepsilon_{ij}^{d}$ and $\sigma_0 = 3K\varepsilon_0$ (B6.177)

2. Show that the principal axes of the stress ad strain tensors coincide.

3. Show that the strain energy density can be expressed as the sum deviatoric and volumetric components,

    $$W(\varepsilon_{ij}) = \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{kk} + \mu \varepsilon_{ij}\varepsilon_{ij} = \frac{9}{2}K\left(\varepsilon_0\right)^2 + \mu \varepsilon_{ij}^{d}\varepsilon_{ij}^{d} = W_p(\varepsilon_{ij}) + W_d(\varepsilon_{ij}). \qquad (B6.178)$$

    where $K = (3\lambda + 2\mu)/3$ is the bulk modulus.

4. Show that the stability condition $W(\varepsilon_{ij}) > 0 \;\; \forall \varepsilon_{ij} \neq 0$ amounts to $K > 0, \mu > 0$.

Solution

See next page

1) The relations (**B**6.175)-(**B**6.176) lead to write

$$\sigma_{ij}^d + \sigma_0 \delta_{ij} = 3\lambda\varepsilon_0\delta_{ij} + 2\mu(\varepsilon_{ij}^d + \varepsilon_0\delta_{ij}) \ . \tag{6.4}$$

Let us recall that the deviatoric tensors have a zero trace

$$tr\, \sigma_{ij}^d = tr\, \varepsilon_{ij}^d = 0 \ .$$

Therefore, computing the trace of (6.4), one obtains

$$3\sigma_0 = 3\lambda\varepsilon_0.3 + 2\mu\varepsilon_0.3$$

and

$$\sigma_0 = 3\lambda\varepsilon_0 + 2\mu\varepsilon_0 = (3\lambda + 2\mu)\varepsilon_0 \ .$$

The definition (**B**6.119)

$$K = \frac{3\lambda + 2\mu}{3}$$

gives

$$\sigma_0 = 3K\varepsilon_0 \ .$$

We rewrite (6.4) successively

$$\begin{aligned} \sigma_{ij}^d + 3K\varepsilon_0\delta_{ij} &= 3\lambda\varepsilon_0\delta_{ij} + 2\mu\varepsilon_{ij}^d + 2\mu\varepsilon_0\delta_{ij} \\ &= (3\lambda + 2\mu)\varepsilon_0\delta_{ij} + 2\mu\varepsilon_{ij}^d \\ &= 3K\varepsilon_0\delta_{ij} + 2\mu\varepsilon_{ij}^d \ . \end{aligned}$$

One finds

$$\sigma_{ij}^d = 2\mu\varepsilon_{ij}^d \ .$$

2) Let us recall that for a second order symmetric tensor $\boldsymbol{L}$, one has $\boldsymbol{Ln} = \lambda\boldsymbol{n}$, where λ is the eigenvalue of $\boldsymbol{L}$ and $\boldsymbol{n}$ the corresponding eigenvector (sec. 1.3.8).
For the deviatoric stress tensor σ_{ij}^d, one has

$$\sigma_{ij}^d n_j = \lambda n_i \ . \tag{6.5}$$

We modify (6.5) as follows

$$\sigma_{ij}^d n_j + \sigma_0 n_i = \sigma_0 n_i + \lambda n_i = (\lambda + \sigma_0) n_i$$

With the help of (**B**6.175), one writes

$$\sigma_{ij}^d n_j + \sigma_0 n_i = \sigma_{ij}^d n_j + \sigma_0\delta_{ij} n_j = (\sigma_{ij}^d + \sigma_0\delta_{ij}) n_j = \sigma_{ij} n_j \ .$$

And we obtain $\sigma_{ij}n_j = (\lambda + \sigma_0)n_i$. This shows that σ_{ij}^d and σ_{ij} have the same eigenvectors.
As regards the displacements, we proceed in a similar fashion. Using (**B**6.177) in (6.5) gives

$$\varepsilon_{ij}^d n_j = \frac{\lambda}{2\mu} n_i \ . \tag{6.6}$$

This shows that σ_{ij}^d and ε_{ij}^d have the same eigenvectors and consequently, the same principal directions. Using (**B**6.176) to rewrite (6.6) leads to the relation

$$(\varepsilon_{ij} - \varepsilon_0\delta_{ij})n_j = \frac{\lambda}{2\mu} n_i$$

or

$$\varepsilon_{ij} n_j = (\varepsilon_0 + \frac{\lambda}{2\mu})n_i \ .$$

Comparing this last relation with (6.6), one concludes that ε_{ij} and ε_{ij}^d have the same eigenvectors. Finally, as
σ_{ij}^d and σ_{ij} have the same eigenvectors n_i,
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ε_{ij} and ε_{ij}^d have the same eigenvectors n_i,
we conclude that ε_{ij} and σ_{ij} have the same eigenvectors n_i and consequently, the same principal directions.

3) The potential strain energy is defined by the next relation

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\varepsilon_{ij}\sigma_{ij} \ .$$

One thus has

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\varepsilon_{ij}\sigma_{ij} = \frac{1}{2}\varepsilon_{ij}(\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}) = \frac{1}{2}\lambda\varepsilon_{kk}^2 + \mu\varepsilon_{ij}\varepsilon_{ij} \ .$$

With the help of (**B**6.176), one writes

$$\begin{aligned} W(\boldsymbol{\varepsilon}) &= \frac{1}{2}\lambda(3\varepsilon_0)^2 + \mu(\varepsilon_{ij}^d + \varepsilon_0\delta_{ij})(\varepsilon_{ij}^d + \varepsilon_0\delta_{ij}) \\ &= \frac{9}{2}\lambda(\varepsilon_0)^2 + \mu(\varepsilon_{ij}^d\varepsilon_{ij}^d + 3\varepsilon_0^2) \\ &= \frac{9}{2}\lambda(\varepsilon_0)^2 + 3\mu(\varepsilon_0)^2 + \mu\varepsilon_{ij}^d\varepsilon_{ij}^d \\ &= \frac{9}{2}\frac{3\lambda + 2\mu}{3}\varepsilon_0^2 + \mu\varepsilon_{ij}^d\varepsilon_{ij}^d \\ &= \frac{9}{2}K\varepsilon_0^2 + \mu\varepsilon_{ij}^d\varepsilon_{ij}^d \ . \qquad (\mathbf{B}6.178) \end{aligned}$$

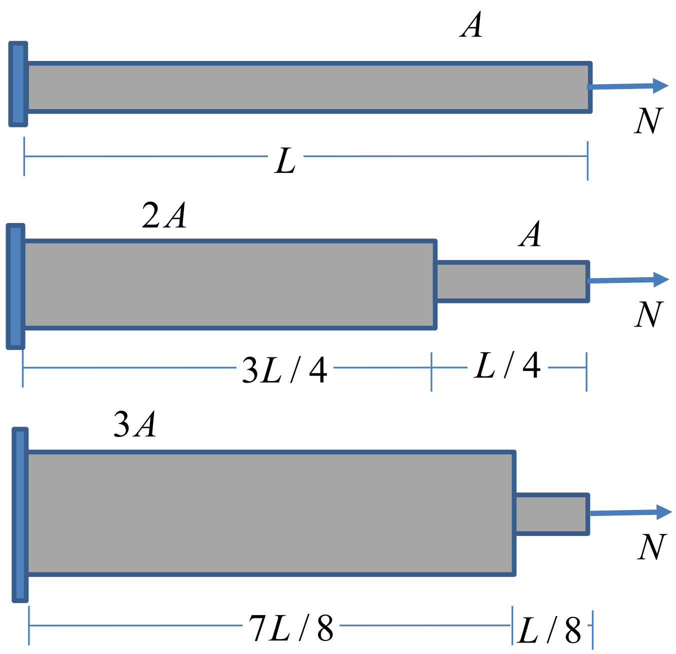
4) For the stability condition

$$W(\boldsymbol{\varepsilon}) > 0$$

to be satisfied, as relation (**B**6.178) is composed of two squares, the coefficients must be such that

$$K > 0 \quad \text{et} \quad \mu > 0 \ .$$

Exercise 3: Three bars of different geometries as shown in the figure below are subjected to the same force *N*. 1. Compare the energies stored in each bar (*A* indicates area and the elastic modulus is known). 2. Calculate the maximum stress in each bar.



Solution:

1. Energies

In the first bar we have $U_1 = \frac{1}{2}\frac{N^2 L}{EA}$

In the second we have $U_2 = \frac{1}{2}\frac{N^2(L/4)}{EA} + \frac{1}{2}\frac{N^2(3L/4)}{E(2A)} = \frac{2+3}{8}\frac{1}{2}\frac{N^2 L}{EA} = \frac{5}{8}U_1$

In the third ne we have $U_2 = \frac{1}{2}\frac{N^2(L/8)}{EA} + \frac{1}{2}\frac{N^2(7L/8)}{E(3A)} = \frac{3+7}{24}\frac{1}{2}\frac{N^2 L}{EA} = \frac{5}{12}U_1$

Thus the larger the volume the smaller the energy for the same force.

2. Maximum Stress (we do not consider stress concentrators in the joints between the different areas of the bars). In each bar the maximum stress is the same and equal to $\sigma_{\max} = \frac{N}{A}$

Exercise 4: A rectangular parallelepiped metallic bloc with dimensions $L_1 = 250$ mm, $L_2 = 200$ mm, $L_3 = 150$ mm is subjected to stresses $\sigma_1 =$ -60 MPa, $\sigma_2 =$ -50 MPa, $\sigma_3 =$ -40 MPa. The mechanical properties are $E = 250$ GPa, $\nu = 0.3$. Calculate: (1) the changes in length L_1, L_2, L_3, (2) the changes in its volume and (3) the strain energy density.

Solution.

We have a three dimensional state of stress without shear stresses. Hook's law gives the strains,

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] = \frac{10^6}{250(10^9)}\left[-60 - 0.3(-50-40))\right] = -132\cdot 10^{-6} = -132\mu\varepsilon$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_3 + \sigma_1)\right] = \frac{10^6}{250(10^9)}\left[-50 - 0.3(-40-60))\right] = -80\mu\varepsilon$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_2 + \sigma_1)\right] = \frac{10^6}{250(10^9)}\left[-40 - 0.3(-50-60))\right] = -28\mu\varepsilon$$

1. $\Delta L_1 = \varepsilon_1 L_1 = -250(132\cdot 10^{-6}) = -0.033mm$
   $\Delta L_2 = \varepsilon_2 L_2 = -200(80\cdot 10^{-6}) = -0.016mm$
   $\Delta L_3 = \varepsilon_3 L_3 = -150(28\cdot 10^{-6}) = -0.0042mm$

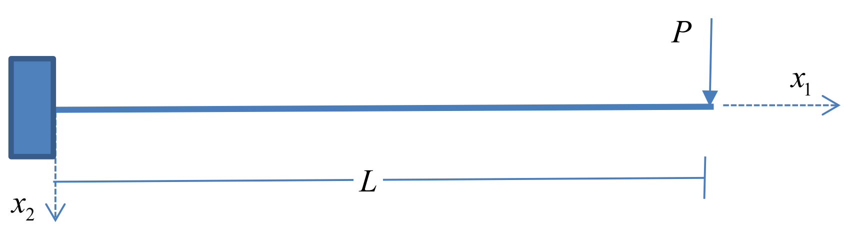
2. $\Delta V = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) L_1 L_2 L_3 = -1800mm^3$

3. $W(\varepsilon_{ij}) = \frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}\sigma_{33}\varepsilon_{33} = 6520\frac{N}{m^2}\frac{m}{m} = J/m^3$

Exercise 5: Use the principle of virtual work to determine the deflection at the free end of the cantilever beam, with known *EI*, shown in the Figure. Use as deflection shape the function and consider only the energy due to bending.

$$u_2(x_1) = \frac{a x_1^2}{2L^3}(3L - x_1) \qquad \text{(a)}$$

where *a* and *b* are constants.



Solution

1. Strain energy,

$$u_2(x_1) = \frac{ax_1^2}{2L^3}(3L - x_1) \Rightarrow \frac{d^2u_2(x_1)}{dx_1^2} = \frac{3a}{L^3}(L - x_1)$$

$$U = \int_0^L \frac{1}{2}\frac{M^2(x_1)}{EI}dx_1 = \frac{EI}{2}\int_0^L \left(\frac{d^2u_2(x_1)}{dx_1^2}\right)^2 dx_1 = \frac{9EIa^2}{2L^6}\int_0^L (L - x_1)^2 dx_1 = \frac{3EIa^2}{2L^3}$$

2. Work of the applied force,

$$\mathbb{W} = P\Delta$$

Where Δ is the deflection at the end-point along *P*.

Principle of virtual work,

$$\delta U = \delta \mathbb{W}$$

From (a) $\Delta = u_2(x_1 = L) = \frac{aL^2}{2L^3}(3L - L) = a$

$$\delta\left(\frac{3EIa^2}{2L^3}\right) = \delta(P\Delta) = \delta(Pa) \Rightarrow \frac{3EI}{2L^3}(2a\delta a) = P\delta a$$

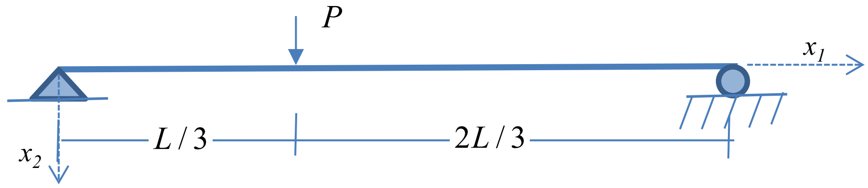
$$\frac{3EI}{2L^3}2a = P \Rightarrow a = \Delta = \frac{PL^3}{3EI}$$

This is the exact value obtained from beam theory.

Exercise 6: A simply supported beam is loaded as shown in the Figure. Determine the deflection at the load application point in the direction of the load. Use the Rayleigh-Ritz method and assume

$$u_2(x_1) = ax_1(L - x_1) \qquad \text{(a)}$$

where a is to be determined. The bending stiffness *EI* is known. Consider only the energy due to bending.



Solution

Potential energy $\Pi = U - W$

$$\Pi = U - P\Delta = \frac{EI}{2}\int_0^L \left(\frac{d^2 u_2(x_1)}{dx_1^2}\right)^2 dx_1 - P\Delta$$

From (a) $\frac{d^2 u_2(x_1)}{dx_1^2} = -2a$.

Define $c = L/3$ and

$$\Delta = u_2(x_1 = c) = ac\left(L - c\right) \qquad \text{(b)}$$

$$\Pi = U - P\Delta = \frac{EI}{2}\int_0^L \left(\frac{d^2 u_2(x_1)}{dx_1^2}\right)^2 dx_1 - Pac\left(L - c\right)$$

$$\Rightarrow \Pi = \frac{EI}{2}(4a^2)L - Pac\left(L - c\right)$$

$$\Rightarrow \frac{\partial \Pi}{\partial a} = 4EIaL - Pc\left(L - c\right) = 0 \qquad \Rightarrow a = \frac{Pc\left(L - c\right)}{4EIL}$$

Use it in (a) and for $x_1 = c$ (or in (b)) to get,

$$\Delta = u_2(x_1 = c) = \frac{Pc^2\left(L - c\right)^2}{4EIL}$$